

Sequences

A **sequence** is a function whose domain is the set of positive integers.

If the domain of $f(x) = 3x - 2$ is limited to positive integers, the sequence 1, 4, 7, 10, 13... is generated.

We write the above sequence using **sequence notation** as $\{a_n\} = \{3n - 2\}$. Here, a is a term in the sequence, and n is the term number. For example, the tenth term in this sequence would be given by $a_{10} = 3(10) - 2 = 28$.

Examples: Write the first five terms of each sequence:

a) $\{s_n\} = \{n^2 + 1\}$

b) $\{c_n\} = \{(-1)^{n+1} \cdot n^2\}$

c) $\{t_n\} = \left\{ \frac{(-1)^n}{(n+1)(n+2)} \right\}$

Note: The term $(-1)^n$ makes the odd terms of a sequence negative and the even terms positive. The terms $(-1)^{n+1}$ or $(-1)^{n-1}$ make the even terms negative and the odd terms positive.

Examples: Write down the n th term of the sequence suggested by each pattern.

a) $-\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \frac{16}{81}, \dots$

b) $1, \frac{1}{2}, 3, \frac{1}{4}, 5, \frac{1}{6}, 7, \frac{1}{8}, \dots$

The Factorial Symbol

If $n \geq 0$ is an integer, then the symbol $n!$ is defined as follows:

$$0! = 1 \quad 1! = 1 \quad 2! = 2 \cdot 1 = 2 \quad 3! = 3 \cdot 2 \cdot 1 = 6 \quad 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$$

$$n! = n \cdot (n-1) \cdot \dots \cdot 3 \cdot 2 \cdot 1$$

$$n! = n \cdot (n-1)!$$

Examples: Evaluate the following:

a) $6!$

b) $\frac{7!}{4!}$

c) $\frac{8!}{6!2!}$

Recursive Formula: A recursively-defined sequence assigns a value to the first term or first few terms, then specifies the n th term by a formula or equation that involves one or more of the previous terms.

Examples: Write the first five terms of each recursively-defined sequence.

a) $a_1 = 3, a_n = 4 - a_{n-1}$

b) $a_1 = -2, a_n = n + 3a_{n-1}$

c) $a_1 = -1, a_2 = 1, a_n = a_{n-2} + na_{n-1}$

d) $a_1 = A, a_n = ra_{n-1}$

Summation Notation

The symbol Σ (the Greek letter sigma) is an instruction to add up the terms of a sequence. The integer k is called the **index** of the sum; it tells you where to start the sum and where to end it. The expression $\sum_{k=1}^n a_k$ is an instruction to add the terms a_k of the sequence $\{a_n\}$ starting with $k = 1$ and ending with $k = n$. It is read, “the sum of a_k from $k = 1$ to $k = n$.”

Examples: Express each sum using summation notation.

a) $1 + 3 + 5 + 7 + \dots + [2(12) - 1]$

b) $\frac{1}{e} + \frac{2}{e^2} + \frac{3}{e^3} + \dots + \frac{n}{e^n}$

Examples: Write out each sum.

a) $\sum_{k=1}^n (k+1)^2$

b) $\sum_{k=0}^{n-1} \left(\frac{1}{3^{k+1}} \right)$

Properties of Sequences

If $\{a_n\}$ and $\{b_n\}$ are two sequences, and c is a real number, then:

- $\sum_{k=1}^n (ca_k) = ca_1 + ca_2 + \dots + ca_n = c(a_1 + a_2 + \dots + a_n) = c \sum_{k=1}^n a_k$
- $\sum_{k=1}^n (a_k + b_k) = \sum_{k=1}^n a_k + \sum_{k=1}^n b_k$
- $\sum_{k=1}^n (a_k - b_k) = \sum_{k=1}^n a_k - \sum_{k=1}^n b_k$
- $\sum_{k=j}^n a_k = \sum_{k=1}^n a_k - \sum_{k=1}^{j-1} a_k$, where $1 < j < n$.

Formulas for Sums of Sequences

- $\sum_{k=1}^n c = \underbrace{c + c + \dots + c}_{n \text{ times}} = cn$
- $\sum_{k=1}^n k = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$
- $\sum_{k=1}^n k^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^n k^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2} \right]^2$

Examples: Find the sum of each sequence.

a) $\sum_{k=1}^5 (5k) =$

b) $\sum_{k=1}^{26} (3k - 7) =$

c) $\sum_{k=1}^{14} (k^2 - 4) =$

d) $\sum_{k=8}^{40} (-3k) =$

e) $\sum_{k=4}^{24} k^3 =$

Arithmetic Sequences

An **arithmetic sequence** is one in which the difference between successive terms of a sequence is always the same number.

An arithmetic sequence may be defined recursively as $a_1 = a$, $a_n = a_{n-1} + d$, where a is the first term and d is the **common difference**. The terms of an arithmetic sequence with first term a_1 and common difference d follow the pattern $a_1, a_1 + d, a_1 + 2d, a_1 + 3d, \dots, a_1 + (n-1)d$, where $d = a_n - a_{n-1}$.

Examples: Determine whether the following sequences are arithmetic:

- a) 3, 7, 11, 15, 19, ... b) $\frac{1}{3}, 1, \frac{5}{3}, \frac{7}{3}, \dots$

★ To show that a sequence is arithmetic, find a_n and a_{n-1} . If $a_n - a_{n-1}$ is a constant (does not have a variable), then the sequence is arithmetic.

Examples: Show that the following sequences are arithmetic and find the common difference.

- a) $\{s_n\} = \{2n - 4\}$ b) $\{b_n\} = \{\ln 2^n\}$

n th Term of an Arithmetic Sequence: For an arithmetic sequence $\{a_n\}$ whose first term is a_1 and whose common difference is d , the n th term is determined by the formula $a_n = a_1 + (n-1)d$.

Examples: Find the n th term and the fifty-first term of the following sequences.

- a) $a_1 = 6, d = -2$ b) $a_1 = 1, d = -1/3$

Examples: Find the indicated term in each arithmetic sequence.

- a) 80th term of -1, 1, 3, ... b) 86th term of $2, \frac{5}{2}, 3, \frac{7}{2}, \dots$

Examples: Find the first term and common difference of the arithmetic sequence described. Give a recursive formula for the sequence, and write a formula for the n th term.

a) 4th term is 3, 20th term is 35

b) 5th term is -2, 13th term is 30

Sum of an Arithmetic Sequence

The sum S_n of the first n terms of an arithmetic sequence $\{a_n\}$ with first term a_1 and common difference d is

given by $S_n = a_1 + a_2 + a_3 + \dots + a_n = \frac{n}{2}(a_1 + a_n)$.

Examples: Find each sum.

a) $-1 + 3 + 7 + \dots + (4n - 5)$

b) $1 + 3 + 5 + \dots + 59$

c) $7 + 1 - 5 - 11 - \dots - 299$

d) $\sum_{n=1}^{90} (3 - 2n)$

e) $\sum_{n=1}^{80} \left(\frac{n}{3} + \frac{1}{2} \right)$

Example: The corner section of a football stadium has 15 seats in the first row and 40 rows in all. Each successive row contains two additional seats. How many seats are in this section?

Geometric Sequences and Geometric Series

A **geometric sequence** is one in which the ratio of successive terms is always the same nonzero number.

A **geometric sequence** may be defined recursively as $a_1 = a$, $a_n = ra_{n-1}$, where a is the first term and $r \neq 0$ is the **common ratio**. The terms of a geometric sequence with first term a_1 and common ratio $r = \frac{a_n}{a_{n-1}}$ follow the pattern $a_1, a_1r, a_1r^2, a_1r^3, \dots, a_1r^{n-1}$.

Examples: Find the common ratio of each geometric sequence and write out the first four terms.

a) $\{b_n\} = \{(-5)^n\}$

b) $\{u_n\} = \left\{ \frac{2^n}{3^{n-1}} \right\}$

★ To determine whether a sequence is arithmetic, geometric, or neither, find a_n and a_{n-1} . If $a_n - a_{n-1}$ is constant, the sequence is arithmetic. If $\frac{a_n}{a_{n-1}}$ is constant, the sequence is geometric.

Examples: Determine whether the given sequence is arithmetic, geometric, or neither. If it is arithmetic, give the common difference. If it is geometric, give the common ratio.

a) $\{5n^2 + 1\}$

b) $\left\{ \left(\frac{5}{4} \right)^n \right\}$

n th Term of a Geometric Sequence: For a geometric sequence $\{a_n\}$ whose first term is a_1 and whose common ratio is r , the n th term is determined by the formula $a_n = a_1r^{n-1}$; $r \neq 0$.

Examples: Find the n th term and the 5th term of the geometric sequence.

a) $a_1 = -2$, $r = 4$

b) $a_1 = 1$, $r = -\frac{1}{3}$

Examples: Find the indicated term of each geometric sequence.

a) 8th term of 1, 3, 9, ...

b) 7th term of 9, -6, 4, ...

Examples: Find the n th term of each geometric sequence.

a) 5, 10, 20, 40, ...

b) $a_2 = 7$, $r = 1/3$

c) $a_3 = 1/3$, $a_6 = 1/81$

Sum of the First n Terms of a Geometric Sequence

The sum S_n of the first n terms of a geometric sequence $\{a_n\}$ with first term a_1 and common ratio r is given

by $S_n = \sum_{k=1}^n a_1 r^{k-1} = a_1 \cdot \frac{1-r^n}{1-r}$, where $r \neq 0, 1$.

Examples: Find each sum.

a) $\frac{3}{9} + \frac{3^2}{9} + \frac{3^3}{9} + \dots + \frac{3^n}{9}$

b) $\sum_{k=1}^{15} 4 \cdot 3^{k-1}$

c) $2 + \frac{6}{5} + \frac{18}{25} + \dots + 2\left(\frac{3}{5}\right)^{15}$

d) $\sum_{n=1}^5 \left(\frac{2}{3}\right)^n$

Infinite Series: An infinite geometric series is the sum of the terms of an infinite geometric sequence. It is

denoted by $\sum_{k=1}^{\infty} a_1 r^{k-1}$. If the sum of the first n terms of the geometric sequence, S_n , approaches a number L as $n \rightarrow \infty$, we say that the infinite geometric series **converges** and that its sum is L . If a series does not converge, it is a **divergent series**.

If $|r| < 1$, the infinite geometric series $\sum_{k=1}^{\infty} a_1 r^{k-1}$ converges. Its sum is $S = \frac{a_1}{1-r}$.

Examples: Determine whether each geometric series converges or diverges. If it converges, find its sum.

a) $2 + \frac{4}{3} + \frac{8}{9} + \dots$

b) $9 + 12 + 16 + \frac{64}{3} + \dots$

c) $1 - \frac{3}{4} + \frac{9}{16} - \frac{27}{64} + \dots$

d) $\sum_{k=1}^{\infty} 3 \left(\frac{3}{2} \right)^{k-1}$

e) $\sum_{k=1}^{\infty} 2 \left(\frac{3}{4} \right)^k$

b. $(3x - 2)^5$

c. $(x^2 + 3y)^3$

d. $(\sqrt{x} - \sqrt{3})^6$

Finding a Particular Coefficient or Term in a Binomial Expansion

Based on the expansion of $(a + b)^n$, the term containing a^j is $\binom{n}{n-j} a^j b^{n-j}$.

Examples:

a. Find the coefficient of x^5 in the expansion of $(x - 1)^8$.

b. Find the coefficient of y^3 in the expansion of $(4y + 2)^6$.

c. Find the fifth term in the expansion of $(2x - y)^9$.

d. Find the third term in the expansion of $(\sqrt{x} + 1)^7$.